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The Canonical Connection in Quantum Mechanics

Péter Lévay

Quantum Theory Group
Institute of Physics
Technical University of Budapest
H-1521 Budapest
Hungary
(e-mail: levay@phy.bme.hu)

David McMullan

School of Mathematics and Statistics
University of Plymouth
Drake Circus, Plymouth, Devon PL4 8AA
U.K.
(e-mail: d.mcmullan@plymouth.ac.uk)

and

Izumi Tsutsui

Institute for Nuclear Study
University of Tokyo
Midori-cho, Tanashi-shi, Tokyo 188
Japan
(e-mail: tsutsui@ins.u-tokyo.ac.jp)

Abstract. In this paper we investigate the form of induced gauge fields that arises in two types of quantum systems. In the first we consider quantum mechanics on coset spaces G/H, and argue that G-invariance is central to the emergence of the H-connection as induced gauge fields in the different quantum sectors. We then demonstrate why the same connection, now giving rise to the non-abelian generalization of Berry's phase, can also be found in systems which have slow variables taking values in such a coset space.

1. Introduction

There are various instances in quantum mechanics when a gauge field appears in a system whose initial formulation did not contain such fields. The most familiar example of this is the emergence of Berry's connection [1] in systems with degeneracies, which leads to a holonomy in energy eigenspaces, *i.e.*, a non-abelian generalization of Berry's phase. Another example is to be found in the different quantum sectors that arise when quantizing on a coset space [2]. For both of these cases, the gauge field that emerges is often found to be of a specific type. Indeed, when the effective configuration space is a coset space G/H, the resulting connection can usually be identified with the so-called H-*connection*, which is a (possibly topological) solution of the Yang-Mills equation on this space. The prime aim of this paper is to clarify why and when this connection arises in these systems.

More precisely, in the context of Berry's phase, the origin of the connection is in some sense obvious from the outset, that is, it comes from the ambiguity in choosing a set of basis vectors in the instantaneous energy eigenspaces. However, what is not obvious and hence remarkable is that in a wide variety of systems of physical interest Berry's connection often (though not always) takes the form of the H-connection [3, 4, 5]. Such systems arise when considering the coupled dynamics of slow and fast variables. In this case we wish to know the form of the connection, also occurring in the Hamiltonian of the effective slow system, in advance. By giving a precise identification of when it is the H-connection the need to calculate energy eigenstates can be avoided.

In constrast, in the context of inequivalent quatizations on coset spaces, the origin of the connection is not quite obvious, and the question is why the specific H-connection can appear at all when quantized. In the account presented in [2] which relies (basically) on Mackey's approach [6], the system of 'free particle' on G/H is considered, where the Hamiltonian is fixed by requiring that there is no operator ordering ambiguity. This is clearly an important criterion, and leads to a system minimally coupled to the H-connection. However, this is not a criterion geometrically motivated, and more importantly, in any attempt at extending these results to field theories such a reliance on a factor ordering argument is unnatural and, indeed, unworkable. What we will show in this paper is that an invariance argument can be developed which highlights the need for such a connection.

The emergence of gauge fields is also recognized recently by a number of other groups [7, 8, 9] using different approaches to quantization. For instance, in [7] spheres S^n embedded in $\mathbb{R}^{n+1}$ are taken as the configuration space and gauge fields are seen to emerge at the quantum level. It will be shown, however, in this paper that these induced gauge fields are none other than the H-connection. This will perhaps support the view that the emergence of gauge fields is not just an artifact of a particular quantization approach but a 'norm' when quantizing on coset spaces.

The plan of this paper is as follows. In Section 2 we will demonstrate how the H-connection emerges in the quantum description of a point particle moving freely on a coset space. In Section 3 we prove that the connection that arises in the quantization scheme of [7] is just the H-connection. In Section 4 the conditions under which Berry's connection reduces to the H-connection will be presented. Section 5 is devoted to our conclusions and discussions.

2. Quantizing on a coset

We begin by arguing that the H-connection — observed by Landsman and Linden [2] in investigating the dynamical aspect of the quantum theory on a coset space G/H — is indeed the natural connection in the quantum system.

Let us first, though, fix our notation (which follows those in [10]). We take G to be a compact Lie group with Lie algebra $\mathfrak{g}$, and H a compact subgroup of G with Lie algebra $\mathfrak{h}$. The Lie algebra $\mathfrak{g}$ has an orthogonal decomposition,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}\,, \tag{2.1}$$

where $\mathfrak{r} = \mathfrak{h}^{\perp}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. This is, in fact, a reductive decomposition, *i.e.,*

$$[\mathfrak{h}, \mathfrak{r}] \subset \mathfrak{r}\,. \tag{2.2}$$

We shall denote bases of the spaces by

$$\begin{aligned} \mathfrak{g} &= \mathrm{span}\{T_m\}\,, & m &= 1, \ldots, \dim \mathrm{G}\,, \\ \mathfrak{h} &= \mathrm{span}\{T_i\}\,, & i &= 1, \ldots, \dim \mathrm{H}\,, \\ \mathfrak{r} &= \mathrm{span}\{T_a\}\,, & a &= 1, \ldots, \dim\,(\mathrm{G/H})\,. \end{aligned} \tag{2.3}$$

Let us recall that in Mackey's account of quantizing on G/H [6] a set of fundamental relations, called a *system of imprimitivity*, is introduced whose irreducible representations give the quantum theories (a full discussion of this can be found in [10]). The upshot of this is that the Hilbert space $\mathcal{H}(\mathrm{G/H})$ on the coset space consists of L^2-functions on G/H belonging to the linear space $\mathcal{H}_\chi$ of some irreducible unitary representation χ of the subgroup H: $\mathcal{H}(\mathrm{G/H}) \simeq L^2(\mathrm{G/H}, \mathcal{H}_\chi)$. Locally, we may take a basis set $\{|q, \chi, \mu\rangle\}$ of the Hilbert space $\mathcal{H}(\mathrm{G/H})$ by

$$|q, \chi, \mu\rangle := |q\rangle \otimes |\chi, \mu\rangle, \tag{2.4}$$

where $|q\rangle$ are the eigenstates in the coordinate representation on G/H and $|\chi, \mu\rangle$ the orthonormal basis vectors in $\mathcal{H}_\chi$. Thus the states in the basis set (2.4) satisfy the orthonormality condition

$$\langle q, \chi, \mu | q', \chi, \nu\rangle = \delta_{\mu\nu}\delta(q - q'), \tag{2.5}$$

with $\delta(q - q')$ being the delta-function on the coset space G/H.

In order to have a singularity-free description we need to introduce a set of patches to cover the coset space G/H. Let $\{U_\alpha\}$ be the local patches introduced, and $\sigma_\alpha : U_\alpha \mapsto \mathrm{G}$ be a continuous section on the patch U_α. On overlaps $U_\alpha \cap U_\beta$ the sections are related by a gauge transformation, namely, for $q \in U_\alpha \cap U_\beta$,

$$\sigma_\beta(q) = \sigma_\alpha(q) h_{\alpha\beta}(q), \tag{2.6}$$

where $h_{\alpha\beta} \in \mathrm{H}$. Accordingly, we consider a sectional basis $\{|q, \chi, \mu\rangle^\alpha\}$ which is a basis set given independently on the patch U_α. Using standard partition of unity arguments, we can define an innerproduct on these and see that all is well defined. The wave functions are then defined to be

$$\psi^\alpha_\mu(q) = {}^\alpha\langle q, \chi, \mu|\psi\rangle \,. \tag{2.7}$$

An important ingredient in Mackey's quantization [6] is that associated with the G-action $q \to g^{-1}q$ for $g \in \mathrm{G}$, which relates any two points on the coset space, there is a corresponding action on the wave functions furnished by the *induced representation*,

$$\big(U(g)\psi\big)^\alpha_\mu(q) = \sum_\nu \pi^\chi_{\mu\nu}\big((\sigma_\alpha(q))^{-1} g \sigma_\beta(g^{-1}q)\big) \ \psi^\beta_\nu(g^{-1}q) \,. \tag{2.8}$$

Here the matrix elements of the unitary operator $\pi^\chi(h)$, implementing the irreducible representation χ, are

$$\pi^\chi_{\mu\nu}(h) := \langle \chi, \mu|\pi^\chi(h)|\chi, \nu\rangle \,, \tag{2.9}$$

and a choice of section has been made on each of the patch, $q \in U_\alpha$ and $g^{-1}q \in U_\beta$. On the sectional basis, this action (2.8) reads

$$U(g)|q, \chi, \mu\rangle^\alpha = \sum_\nu |gq, \chi, \nu\rangle^\beta \, \pi^\chi_{\nu\mu}\big((\sigma_\beta(gq))^{-1} g \sigma_\alpha(q)\big) \ , \tag{2.10}$$

where we put $g \to g^{-1}$ for later convenience. In effect, the induced representation (2.10) consists of a rotation in the space $\mathcal{H}_\chi$ and a translation in the coset space G/H, both determined by g and q. Using the naturally defined measure on the coset space G/H, one can readily show that (2.10) indeed provides a unitary representation of G [10].

Now we shall consider the quantum mechanics of a point particle moving freely on the coset space G/H. Here the term 'free' is meant to indicate that the system under consideration is *homogeneous* over G/H, and that the dynamics of the particle is that of a free particle when observed locally. Note that in order to get the Schrödinger equation

for the wave functions (2.7) describing the point particle of this system, we need to use the G-action to ensure the homogeneity (as it is the only means available on G/H for this purpose). But since the G-action on the wave functions (2.8) is section dependent, we need a covariant derivative (with respect to q) such that the section dependence disappears in the physical dynamics.

To be explicit, let us consider the state

$$|\chi, \bar{\mu}\rangle := \sum_{\nu} |e, \chi, \nu\rangle^{\beta}\, \pi_{\nu\mu}\big((\sigma_{\beta}(e))^{-1}\big) \ , \tag{2.11}$$

where e is the identity point in the coset G/H. Then (2.10) allows us to write the basis states at q as

$$|q, \chi, \mu\rangle^{\alpha} = U\big(\sigma_{\alpha}(q)\big)|\chi, \bar{\mu}\rangle \ , \tag{2.12}$$

which shows that the G-action allows for obtaining all the basis states over G/H by the unitary G-action from the reference state (2.11). It is then easy to see from (2.10) that, under the change of section (2.6), the basis states undergo the rotation,

$$|q, \chi, \mu\rangle^{\alpha} \to |q, \chi, \mu\rangle^{\beta} = U(\sigma_{\alpha} h_{\alpha\beta})|\chi, \bar{\mu}\rangle = \sum_{\nu} |q, \chi, \nu\rangle^{\alpha} \pi^{\chi}_{\nu\mu}(h_{\alpha\beta}) \ . \tag{2.13}$$

Thus, the connection used in the covariant derivative must compensate the derivative factor in the Schrödinger equation arising from the rotation in (2.13). Actually, in the theory of vector bundles associated with the principal bundle G(G/H, H), the term ‘connection’ already implies this property. This, however, is not enough to single out the connection relevant to our system on G/H.

The crucial point in specifying the connection is the homogeneity over the coset G/H mentioned above. We note that for the system to be homogeneous the connection must also be homogeneous physically, that is, it must be invariant under the G-action up to a gauge transformation of the group H (*i.e.,* up to a change of section). In other words, the curvature of the connection is constant over G/H. Now the theory of invariant connections (see Theorem 11.1 on p.103 of Ref.[11]) asserts that such a connection is always given by the H-connection $A^{\rm H} := \sigma_{\alpha}^{-1}(q) d\sigma_{\alpha}(q)|_{\mathfrak{h}}$, which is the (pullback of the) canonical 1-form projected down to the subspace $\mathfrak{h} \subset \mathfrak{g}$. In the present context, the invariant connection that arises in the covariant derivative acting on the wave functions (2.7) is the H-connection in the representation χ:

$$\sum_{i} A^{\rm H}_{i}(q)(T_{i})_{\mu\nu} = \langle \chi, \bar{\mu}|\ U^{-1}\big((\sigma_{\alpha}(q)\big)\, dU\big(\sigma_{\alpha}(q)\big)|_{\mathfrak{h}}\, |\chi, \bar{\nu}\rangle \ . \tag{2.14}$$

One can readily confirm that its curvature is indeed constant over G/H and that it does transform as a connection under the change of section (2.6).

In short, we see that the covariant derivative used for the Schrödinger equation must contain the H-connection in the form (2.14), if we are to consider the homogeneous free particle system over G/H requiring the independence of the choice of section. This G-invariance is, we feel, more fundamental than the factor ordering criterion adopted in [2]. However, for completeness, we now need to see what form of Hamiltonian comes out of our analysis.

To begin with, let us note that our vector-valued wave functions $\psi^{\alpha}_{\mu}(q)$, provided by the irreducible representation χ of H , may be expanded in terms of the 'harmonics' $U^{\Lambda}_{\xi}\big(\sigma^{-1}_{\alpha}(q)\big)$ over the coset space G/H [12],

$$\psi^{\alpha}_{\mu}(q) = \sum_{\Lambda}\sum_{\rho,\xi} c^{\Lambda}_{\rho\xi}\, U^{\Lambda}_{\xi}\big(\sigma^{-1}_{\alpha}(q)\big)_{\mu\rho} \,. \tag{2.15}$$

In this expansion ξ is the index of multiplicity of the representation χ appearing in the irreducible representation Λ of G upon restriction to H, and the range of ρ equals the dimension of the representation Λ.

We recall that the Frobenius reciprocity theorem tells that in the above summation only those Λ of G occur which contain the representation χ of H when restricted to the subgroup. (For brevity we henceforth omit α which labels the patch to which the point q belongs.) But the message important to us here is that we can now work with the section variable $\sigma^{-1}(q)$ instead of the cordinates q on the coset space. We shall for the sake of simplicity consider the principal bundle G(G/H, H) first. Because our vector bundle in question is the associated bundle via the irreducible representation of H, the covariant derivative in the vector bundle will follow immediately from that of the principal bundle.

Consider now the vector fields X_m defined by the relation,

$$X_m\sigma^{-1}(q) = \sigma^{-1}(q)T_m \,. \tag{2.16}$$

These vector fields are just the generalizations of the usual Killing vector fields regarded as first order differential operators. (In the context of Berry's phase these are modified symmetry generators for effective Hamiltonians [5].) The fact that such vector fields do exist can be seen explicitly by examining the infinitesimal version of $g\sigma(q) = \sigma(gq)h(g, q)$, which

leads to a first order differential operators for X_m satisfying the commutation relations of the Lie algebra $\mathfrak{g}$.

We shall then consider the following covariant derivative

$$\nabla_m := -\mathcal{D}_m^n(\sigma^{-1})X_n \ , \tag{2.17}$$

where $\mathcal{D}_m^n(\sigma)$ is the 'adjoint matrix' (the matrix of the adjoint representation of G in the basis T_m) defined by

$$\sigma^{-1}(q)T_m\sigma(q) = \mathcal{D}_m^n(\sigma)T_n \ . \tag{2.18}$$

Using this, we may invert (2.16) to get

$$\mathcal{D}_n^m(\sigma^{-1})X_m\sigma^{-1} = T_n\sigma^{-1} \ . \tag{2.19}$$

We hence find that our covariant derivative (2.17) satisfies

$$\nabla_m\sigma^{-1} = -T_m\sigma^{-1} \ , \tag{2.20}$$

that is, it behaves just as $-T_m$ on σ^{-1}.

The $\mathfrak{r}$ component of ∇_m is the covariant derivative with respect to the H-connection. To see this, following the standard line of argument [13] one decomposes the canonical 1-form as $\sigma^{-1}d\sigma = -d(\sigma^{-1})\sigma = A^{\mathrm{H}} + e$ where $A^{\mathrm{H}} = A^i_\alpha T_i dq^\alpha$ is the H-connection and $e = \sigma^{-1}d\sigma|_{\mathfrak{r}} = e_\alpha{}^a T_a dq^\alpha$ is the vielbein, with $T_i \in \mathfrak{h}$ and $T_r \in \mathfrak{r}$ in the orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$. Using the inverse of the vielbein, $e^\alpha{}_a e_\alpha{}^b = \delta^b_a$, one may cast the canonical 1-form into the vielbein frame. This yields

$$(\partial_a + e_a{}^\alpha A^i_\alpha T_i)\sigma^{-1} = -T_a\sigma^{-1} = \nabla_a\sigma^{-1} \ , \tag{2.21}$$

where (2.20) is used in the last equality, proving therefore our claim.

When we go over to the vector bundle from the principal bundle, we have to act with the covariant derivative on the expansion (2.15), hence we are to use the particular representation χ for the generators T_i of $\mathfrak{h}$. It is then clear that the covariant derivative acts in an extremely simple manner on the wave functions. In fact, the property (2.20) shows that the covariant derivative in the representation χ is indeed the representation of the element T_m on such wave functions. Hence, if we adopt for the Hamiltonian the quadratic Casimir $X_mX^m = \nabla_m\nabla^m$ of the group G — which is G-invariant by construction — we find that the Hamiltonian is given by the square of the covariant derivative $\nabla_a\nabla^a$ modulo a constant which is the value of the quadratic Casimir of the subgroup H evaluated on the irreducible representation χ. Thus the free, homogeneous Hamiltonian given by the quadratic Casimir leads precisely to the Hamiltonian for the particle minimally coupled to the H-connection, that is, the Hamiltonian argued by Landsman and Linden [2].

3. Quantizing on an n-Sphere

In the approach to quantizating on spheres S^n proposed by Ohnuki and Kitakado [7] there appeared (possibly topological) gauge fields on the spheres as a result of inequivalent quantizations. These (infinitely) many inequivalent quatizations are labelled by the irreducible representations of the group $SO(n)$ — an important feature shared with Mackey's approach [6] where one regards S^n as $SO(n+1)/SO(n)$. Thus it would be natural to expect that the gauge fields observed in [7] may coincide with the H-connection found by Landsman and Linden [2] in Mackey's approach. We shall show below that this is indeed the case.

But let us first recall the quantization and the gauge fields discussed in [7]. There, quantization is prescribed by embedding the sphere S^n in $\mathbb{R}^{n+1}$ and then postulating a 'fundamental algebra' as a set of quantum relations, generalizing the conventional canonical commutation relations. The fundamental algebra is the Lie algebra of $E(n+1)$, the Euclidean group in $n+1$ dimensions given by the semidirect product of $SO(n+1)$ and $\mathbb{R}^{n+1}$, and finding the Hilbert space $\mathcal{H}(S^n)$ amounts to finding the representations of the group taking into account the constraint that restricts to the sphere. Wigner's technique then allows for constructing explicitly the representations of $E(n+1)$ from the irreducible representations of the subgroup $SO(n)$, which is the isometry group of $SO(n+1)$ acting on S^n. According to this, the representations (of the Lie algebra) of $E(n+1)$ may be found by looking at the infinitesimal generators of the Wigner rotation. In Mackey's language the Wigner rotation corresponds to the matrix element[1]

$$Q_{\mu\nu}(g,q) := \pi^{\chi}_{\mu\nu}\big((\sigma(q))^{-1} g \sigma(g^{-1}q)\big) \ , \tag{3.1}$$

representing the rotations in the components of the vector-valued wave function in the induced representation (2.8). In the present case $g \in SO(n+1)$ and q stands for a vector on the sphere S^n embedded in $\mathbb{R}^{n+1}$, and we take the radius of the sphere to be unity, $\sum_{\alpha=1}^{n+1}(q^\alpha)^2 = 1$. In this embedding we adopt the convention that any function on S^n is smoothly extended to $\mathbb{R}^{n+1}$ by continuing the value of the function constantly along the direction of the radius. This implies that any function $f(q)$ defined this way obeys the condition, $q^\alpha \partial_\alpha f(q) = 0$, where $\partial_\alpha = \partial/\partial q^\alpha$.

[1] We here assume for simplicity that q and $g^{-1}q$ are in the same patch where a single section σ is available.

We label the basis of the Lie algebra $so(n+1)$ by antisymmetric operators $T_{\alpha\beta}$ with α and β running over $1, \ldots, n+1$. The $so(n)$ subalgebra is identified with the generators T_{ab}, where a and b can take values $1, \ldots, n$. The reductive decomposition $so(n+1) = so(n) \oplus \mathfrak{r}$ is then given by $so(n+1) = \text{span}\{T_{ab}\} \oplus \text{span}\{T_a\}$ where $T_a = T_{a,n+1}$. The commutation relations are then

$$
\begin{aligned}
[T_{ab}, T_{cd}] &= \delta_{ad}T_{bc} + \delta_{bc}T_{ad} - \delta_{ac}T_{bd} - \delta_{bd}T_{ac} \ , \\
[T_{ab}, T_c] &= \delta_{bc}T_a - \delta_{ac}T_b \ , \\
[T_a, T_b] &= -T_{ab} \ .
\end{aligned}
\tag{3.2}
$$

To make the presentation easier we now omit the label π^χ for the representation used.

Corresponding to the infinitesimal transformation $g = e^{\frac{1}{2}\epsilon_{\alpha\beta}T_{\alpha\beta}} = 1 + \frac{1}{2}\epsilon_{\alpha\beta}T_{\alpha\beta}$ with $\epsilon_{\alpha\beta}$ being real antisymmetric parameters, we have the Wigner rotation,

$$
Q(g, q) = 1 + \tfrac{1}{2}\epsilon_{\alpha\beta}f_{\alpha\beta}(q) \ , \tag{3.3}
$$

where $f_{\alpha\beta}(q)$ are the generators of the rotation. Then, the combination [7]

$$
A_\alpha(q) := f_{\alpha\beta}(q)\, q^\beta \ , \tag{3.4}
$$

is seen to appear in the Hamiltonian in the form covariantly coupled to a particle, and hence is regarded as an induced gauge field. We now show that this gauge field (3.4) is in fact the H-connection.

To this end, observe first that from (3.1) the generators in (3.3) are given by

$$
f_{\alpha\beta}(q) = \sigma^{-1}(q)T_{\alpha\beta}\sigma(q) - \sigma^{-1}(q)\partial_\mu\sigma(q)\frac{\partial q^\mu(\epsilon)}{\partial\epsilon_{\alpha\beta}}\Big|_{\epsilon=0} \ , \tag{3.5}
$$

where $q^\mu(\epsilon) := (g\,q)^\mu = q^\mu + \frac{1}{2}\epsilon_{\alpha\beta}(T^{\text{def}}_{\alpha\beta})_{\mu\nu}q^\nu$, and $(T^{\text{def}}_{\alpha\beta})_{\mu\nu} = \delta_{\alpha\mu}\delta_{\alpha\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}$ is the defining representation of $so(n+1)$. From this we get

$$
\frac{\partial q^\mu(\epsilon)}{\partial\epsilon_{\alpha\beta}}\Big|_{\epsilon=0} = \delta^{\alpha\mu}q^\beta - \delta^{\beta\mu}q^\alpha \, . \tag{3.6}
$$

It is then easy to see that under the change of section $\sigma(q) \to \sigma(q)h(q)$ for some $h(q) \in SO(n)$ the gauge field (3.4) transforms as a connection,

$$
A_\alpha(q) \to h^{-1}(q)A_\alpha(q)h(q) - h^{-1}(q)\partial_\alpha h(q) \ . \tag{3.7}
$$

This is also evident from the expression,

$$A_\alpha(q) = \sigma^{-1}(q) T_{\alpha\beta} q^\beta \sigma(q) - \sigma^{-1}(q)\partial_\alpha \sigma(q) \ , \tag{3.8}$$

obtained from the definition (3.4).

Consider now the section

$$\sigma(q) = e^{\theta^a(q) T_a} \ , \tag{3.9}$$

which provides a local mapping from S^n to G $= SO(n+1)$. The inverse mapping is given by

$$q^a := \theta^a \frac{\sin|\theta|}{|\theta|}, \quad a = 1, \ldots, n, \qquad q^{n+1} := \cos|\theta| \,, \tag{3.10}$$

where $\theta^t = (\theta^1, \ldots, \theta^n)$ and $|\theta| = \sqrt{\theta^t\theta} = \sqrt{\sum_a (\theta^a)^2}$. With the section (3.9) one finds that the relevant parts of the adjoint matrix (2.18),

$$\begin{aligned} \sigma^{-1}(q) T_{ab} \sigma(q) &= \mathcal{D}_{ab}^{cd}\, T_{cd} + \mathcal{D}_{ab}^{c}\, T_c \ , \\ \sigma^{-1}(q) T_a \sigma(q) &= \mathcal{D}_a^{bc}\, T_{bc} + \mathcal{D}_a^{b}\, T_b \ , \end{aligned} \tag{3.11}$$

take the form [5]

$$\mathcal{D}_a^{bc} = \tfrac{1}{2}(q^b \delta_a^c - q^c \delta_a^b) \,, \tag{3.12}$$

and

$$\mathcal{D}_{ab}^{cd} = \tfrac{1}{2}(\delta_a^c \delta_b^d - \delta_b^c \delta_a^d) + \frac{q_b(q^c\delta_a^d - q^d\delta_a^c) + q_a(q^d\delta_b^c - q^c\delta_b^d)}{2(1+q^{n+1})} \ . \tag{3.13}$$

To show that (3.8) is the H-connection we note that the $\mathfrak{h}$-part in the first term on the right hand side of (3.8) vanishes,

$$\sigma^{-1}(q)\, T_{\alpha\beta} q^\beta \sigma(q)|_{\mathfrak{h}} = 0 \ . \tag{3.14}$$

For $\alpha = n+1$, this is obvious since the middle piece $T_{\alpha\beta}q^\beta$ that is conjugated under $\sigma(q)$ is precisely proportional to the argument in the exponential of $\sigma(q)$; see (3.9) and (3.10). For $\alpha = a \neq n+1$, using (3.12), (3.13) and the antisymmetry of T_{cd}, we have

$$\sigma^{-1}(q) T_{a\beta} q^\beta \sigma(q)|_{\mathfrak{h}} = (q^b \mathcal{D}_{ab}^{cd} + q^{n+1} \mathcal{D}_a^{cd}) T_{cd} = 0 \,, \tag{3.15}$$

which establishes (3.14).

Now since the gauge field (3.8) must lie anyway in the space $\mathfrak{h} = so(n)$ by construction (because it is formed out of the generators of the $SO(n)$ Wigner rotation), we see that the

$\mathfrak{r}$-part of the two terms in the right hand side of (3.8) must precisely cancel each other. Combined with (3.14), this implies that

$$A_\alpha(q) = -\sigma^{-1}(q)\partial_\alpha\sigma(q)|_{\mathfrak{h}} \ , \tag{3.16}$$

that is, Ohnuki-Kitakato's gauge field (3.4) is in fact the H-connection (up to the irrelevant sign). In terms of the section (3.9) the H-connection reads[2]

$$\sigma^{-1}(q)d\sigma(q)|_{\mathfrak{h}} = \frac{1}{1+q^{n+1}}\sum_{a,b}^{n} q_a dq_b \, T_{ab} \ , \tag{3.17}$$

which of course agrees with the expression found in [7].

In passing, we mention that in a recent paper [15] it is pointed out that the gauge field (3.4) can be mapped into the 'generalized BPST instanton' solution found earlier [16] — a solution of the Yang-Mills equation on S^n which is topologically nontrivial for n even and trivial for n odd. The above result implies that this solution is essentially identical to the H-connection, although the meaning of self-duality can change under the mapping.

[2] The confirmation by a direct computation is also given in [14].

4. Berry's connection as the H-connection

Berry's phase arises in systems where the Hamiltonian has degenerate eigenstates labelled by a collection of parameters, which are identified with the slow degrees of freedom. Adiabatically decoupling the fast variables from these slow ones results in an effective theory with a gauge structure in the slowly varying system [1]. The form of the gauge field that emerges is governed by the geometry of the slow system. In applications the degeneracies reflect a symmetry of the system, hence the slow system is usually identified with a coset space G/H. Such an identification emerges from a Hamiltonian of the form

$$H(q) = U(q) H_0 U^{-1}(q)\,, \tag{4.1}$$

where $q \in \mathrm{G/H}$ are the slow variables, $U(g)$ is a unitary irreducible representation of G and H_0 is typically an element of the enveloping algebra of the subgroup H, commuting with the restriction of the representation U to H. It is readily confirmed [3, 4, 5] that if we let $U(q)$ be in the form $U(\sigma(q))$, then (2.12) furnishes the eigenstates of the Hamiltonian with $|\chi, \bar{\mu}\rangle$ being the eigenstates of H_0 labelled by some irreducible representation χ of H. Thus our representation χ is obtained from the given representation of G by restriction to H, followed by a further restriction to an invariant subspace. Using the states (2.12) (again dropping the label α for the patch to which q belongs) Berry's connection reads

$$\begin{aligned}\sum_m A_m^{\mathrm{Berry}}(q)(T_m)_{\mu\nu} &= \langle q, \chi, \mu | d | q, \chi, \nu \rangle \\ &= \langle \chi, \bar{\mu} |\, U^{-1}\big(\sigma(q)\big)\, dU\big(\sigma(q)\big)\, | \chi, \bar{\nu} \rangle \\ &= \sum_i A_i^{\mathrm{H}}(q)(T_i)_{\mu\nu} + \sum_a e_a(q) \langle \chi, \bar{\mu} | U(T_a) | \chi, \bar{\nu} \rangle\end{aligned} \tag{4.2}$$

The identification of this connection with the H-connection clearly depends on whether the final term is zero or not. In applications this term is often set equal to zero by hand [4, 5]. That this term is not always zero, though, is best seen through an explicit example.

Consider the situation where the slow variables parametrize a three sphere S^3, now viewed as the coset space $SO(4)/SO(3)$. This would arise, for example, from (4.1) by taking H_0 to be the quadratic Casimir for $SO(3)$. In the Lie algebra of $SO(4)$ we take the reductive decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r} = \mathrm{span}\{T_i\} \oplus \mathrm{span}\{T_a\}\,, \qquad i,\, a = 1,\, 2,\, 3\,. \tag{4.3}$$

with the T_i's forming an $su(2)$ algebra, $[T_i, T_j] = \varepsilon_{ijk} T_k$, and the remaining commutators being

$$[T_i, T_a] = \varepsilon_{iab} T_b \ , \tag{4.4}$$

and

$$[T_a, T_b] = \varepsilon_{abi} T_i \, . \tag{4.5}$$

The non H-connection part of Berry's connection is, in this example, $\sum_a e_a \langle jm|T_a|jm'\rangle$, where we have reverted to the familiar notation for the representation of angular momentum. We now show that the matrix element $\langle jm|T_a|jm'\rangle$ need not be zero in general.

For this, we note first that the commutator (4.4) implies that the basis vectors in $\mathfrak{r}$ transform as a vector (spin 1) operator. To emphasise this fact we will, henceforth, denote these operators by $T_a^{(1)}$. The Wigner-Eckart theorem then tells us that the m, m' and a dependence of this matrix element resides in the Clebsch-Gordan coefficients $\langle jm'1a|jm\rangle$:

$$\langle jm|T_a^{(1)}|jm'\rangle = \langle jm'1a|jm\rangle \langle j\|T^{(1)}\|j\rangle \ , \tag{4.6}$$

where $\langle j\|T^{(1)}\|j\rangle$ is the reduced matrix element which is independent of m, m' and a. In terms of the basis $T_\pm := i(T_1 \pm iT_2)$, $T_0 := iT_3$, the Clebsch-Gordan coefficients are given by

$$\langle jm'1a|jm\rangle = \frac{\delta_{m'+a,m}}{\sqrt{j(j+1)}} \begin{cases} \mp\sqrt{(j\pm m)(j\mp m+1)/2}, & \text{if } a = \pm; \\ m, & \text{if } a = 0. \end{cases} \tag{4.7}$$

Upon identifying the same a and i, we find that these coefficients are related to the representation matrix elements $\langle jm'|T_i|jm\rangle$ of the $su(2)$ generators T_i, $i = +, -, 0$. This allows us to rewrite (4.6) as

$$\langle jm|T_a^{(1)}|jm'\rangle = a_j \langle jm'|T_i|jm\rangle \ , \tag{4.8}$$

where the prefactor a_j is

$$a_j = -\frac{\langle j\|T^{(1)}\|j\rangle}{\sqrt{j(j+1)}} \, . \tag{4.9}$$

We recall that the action of any vector operator on the state $|jm'\rangle$ is determined by two reduced matrix elements. For $T^{(1)}$ these are a_j and the reduced matrix element $\langle j-1\|T^{(1)}\|j\rangle$. However, the action of $T^{(1)}$ is also fixed by the fact that it comes from a representation of $SO(4)$. Exploiting these two facts allows us to determine the allowed values for a_j.

The irreducible unitary representations of $SO(4)$ are labelled by two numbers (k_0, c), where $k_0 = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and $c = \pm(k_1 + 1)$ with $k_1 = k_0, k_0 + 1, k_0 + 2, \ldots$. (For a clear account of this see [17].) The representation space is then decomposed into the direct sum

$$\mathcal{R}(k_0, c) = \bigoplus_{j=k_0}^{k_1} \mathcal{R}^j \,, \tag{4.10}$$

of the irreducible representations $\mathcal{R}^j$ of $SO(3)$ spanned by the angular momentum states $|jm\rangle$, $m = -j, \ldots, j$. In such a representation one finds that a_j is given by

$$a_j = \frac{k_0 c}{j(j+1)} \,. \tag{4.11}$$

From this we deduce that if $k_0 \neq 0$ then Berry's connection does not correspond to the H-connection.

This example can be extended to more general coset spaces in much the same way by using the generalized Wigner-Eckart theorem (see, for example, [18]). The conclusion reached is that, in general, Berry's connection is not the H-connection. The question we now want to address is what additional structures are needed in order to ensure that they do coincide. To motivate our analysis of this problem it is again useful to return to the three sphere example discussed above.

From (4.11) we see that the relevant reduced matrix element vanishes only when $k_0 = 0$. In this case (and only in this case) the representations $(k_0 = 0, c)$ and $(k_0 = 0, -c)$ of $SO(4)$ are unitarily equivalent (there is no parity doubling [17]). The representation space becomes the direct sum

$$\mathcal{R}(0, n) = \bigoplus_{j=0}^{n-1} \mathcal{R}^j \,, \qquad \text{where} \quad n = 1, 2, \ldots. \tag{4.12}$$

The action of T_i on $\mathcal{R}^j$ is the standard one, changing the value of m by ± 1. From (4.4) one can also show that the action of $T_a^{(1)}$ on $\mathcal{R}^j$ changes the value of j by ± 1. Thus the state $|jm\rangle = |n-1, n-1\rangle$ is both a highest weight vector for the irreducible representation on $\mathcal{R}^{n-1}$ of $SO(3)$, and for the irreducible representation on $\mathcal{R}(0, n)$ of $SO(4)$. This cannot hold for any of the other $(k_0 \neq 0)$ representations of $SO(4)$ since the parity doubling found in those representations would then imply that such a vector was a highest weight for two inequivalent representations.

We shall use this example as a motivation for the following restriction on the allowed states $|\chi, \bar{\mu}\rangle$ that occur in (4.2). Recall first that by definition the reference basis states satisfy

$$U(h)|\chi, \bar{\mu}\rangle = \sum_{\nu} |\chi, \bar{\nu}\rangle \pi^{\chi}_{\nu\mu}(h), \qquad \text{for} \quad h \in \mathrm{H}\,. \tag{4.13}$$

Let Λ be the highest weight labelling the representation of the group G in question. We shall consider the *highest subspace* $\mathcal{H}_\Lambda$, which is the subspace of the representation space $\mathcal{H}$ of G realizing (4.13) and also contains the vector $|\Lambda\rangle$ corresponding to the highest weight. We then claim that, for a wide class of systems, by choosing the subspace as a highest subspace we will manage to obtain merely the $\mathfrak{h}$-part of Berry's connection. To prove this it is convenient to develop an alternative description of the highest subspace $\mathcal{H}_\Lambda$.

For this, let us restrict ourselves to cosets G/H, where the subgroup H is given by the centralizer S_K of some element $K \in \mathfrak{g}$. This corresponds to the Hamiltonian (4.1) whose parameter space is the coadjoint orbit of the group G passing through K discussed in [3, 5]. If K is a regular semisimple element [19] of $\mathfrak{g}$ then H in this case is just the Cartan subgroup T regarded as the maximal torus containing K, but if not then H is greater than T. Let Σ be the root system of G relative to T, and let Σ_K be the root system of H relative to T. By considering the complexification $\mathfrak{g}_{\mathrm{c}}$ of $\mathfrak{g}$ we have the Cartan decomposition

$$\mathfrak{g}_{\mathrm{c}} = \mathfrak{t}_{\mathrm{c}} \oplus \sum\nolimits_{\alpha\in\Sigma} \mathfrak{g}_\alpha \ , \tag{4.14}$$

where $\mathfrak{t}_{\mathrm{c}}$ is the complexification of the Cartan subalgebra $\mathfrak{t}$, and $\mathfrak{g}_\alpha$ is the root space corresponding to the root α. Similarly we have

$$\mathfrak{h}_{\mathrm{c}} = \mathfrak{t}_{\mathrm{c}} \oplus \sum\nolimits_{\alpha\in\Sigma_K} \mathfrak{h}_\alpha \ . \tag{4.15}$$

Next, let W be a Weyl chamber of $\mathfrak{t}$ relative to G, and W_K be a Weyl chamber of $\mathfrak{t}$ relative to H. We can define the positive roots Σ^+ (Σ^+_K) of Σ (Σ_K) with respect to W (W_K). It is then guaranteed [20] that there exists a 'K admissible Weyl chamber' satisfying: (i) $\Sigma^+ \cap \Sigma_K = \Sigma^+_K$, and (ii) if $\alpha \in \Sigma^+ - \Sigma^+_K$, $\beta \in \Sigma_K$ and $\alpha + \beta \in \Sigma$, then $\alpha + \beta \in \Sigma - \Sigma^+_K$.

Armed with this, we then show that the highest subspace $\mathcal{H}_\Lambda$ can alternatively be characterized by

$$\mathcal{H}_\Lambda = \left\{ \, |\phi\rangle \in \mathcal{H} \, \Big| \; U(T_\alpha)|\phi\rangle = 0, \quad \forall \alpha \in \Sigma^+ - \Sigma^+_K \right\} \, . \tag{4.16}$$

Note first that the states defined by (4.16) are invariant under the action of H in $\mathcal{H}$. Indeed, for those generators of $\mathfrak{h}$ belonging to the Cartan subalgebra $\mathfrak{t}$ this is obvious since for $T_i \in \mathfrak{t}$, $[T_i, T_\alpha] = \alpha(T_i)T_\alpha$. If T_β is a generator of $\mathfrak{h}$ not in $\mathfrak{t}$ then, using the reductivity of the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$, we get, for $\alpha \in \Sigma^+ - \Sigma_K^+$,

$$U(T_\alpha)U(T_\beta)|\phi\rangle = U(T_\beta)U(T_\alpha)|\phi\rangle + U([T_\alpha, T_\beta])|\phi\rangle = C_{\alpha\beta}^{\alpha+\beta} U(T_{\alpha+\beta})|\phi\rangle \ , \tag{4.17}$$

which vanishes since $\alpha + \beta \in \Sigma^+ - \Sigma_K^+$.

Second, the unitary action in (4.16) is also irreducible. To see this, suppose that it is reducible. Then there exists some $|\Omega\rangle \neq |\Lambda\rangle$ for which

$$U(T_\beta)|\Omega\rangle = 0, \qquad \text{for} \quad \beta \in \Sigma_K^+ \ , \tag{4.18}$$

i.e., $U(T_\beta)$ is a step operator in $\mathfrak{h}$ annihilating this state. It then follows that both the operators $U(T_\alpha)$ and $U(T_\beta)$, where $\alpha \in \Sigma^+ - \Sigma_K^+$ and $\beta \in \Sigma_K^+$, annihilate $|\Omega\rangle$. Hence this state is annihilated by any $U(T_\alpha)$ for $\alpha \in \Sigma^+$, which implies that $|\Omega\rangle$ is a highest weight. But since we cannot have two highest weights, we see that $\mathcal{H}_\Lambda$ defined by (4.16) is irreducible and hence must be the highest subspace satisfying (4.13).

Having established (4.16), we now find, for such highest subspace states,

$$\sum_{\alpha \in \Sigma - \Sigma_K} e_\alpha(q) \langle \chi, \bar{\mu} | U(T_\alpha) | \chi, \bar{\nu} \rangle = 0 \ , \tag{4.19}$$

on account of $T_{-\alpha} = {T_\alpha}^\dagger$. Clearly, then, we can conclude that if a highest subspace is used in the construction of Berry's connection, then there will be no $\mathfrak{r}$-part and hence it will be the H-connection — the claim we wished to prove.

5. Conclusions and Discussions

In this paper we have argued that the induced connection that appears on a coset space in Mackey's quantization scheme admits a natural interpretation, that is, it arises from the homogeniety criterion required for the Hamiltonian. This led to an alternative account from [2] of why the Hamiltonian on G/H involves the induced H-connection. Being geometrical, our criterion will be useful even in other quantization approaches and, possibly, in attempts at extending the quantization scheme to field theories. Indeed, we have shown that the gauge field induced in a slightly different approach [7] is again the H-connection — a fact suggesting a universal feature of the quantum theory on such topologically non-trivial spaces. In connection with this, it is worth mentioning that even in the 'confining approach' [21] to quantization, which is totally different from Mackey's approach, one can still observe an induced gauge field which also appears to be of the type of the H-connection [8, 9].

The appearance of the H-connection in the other context — Berry's phase — was then analyzed in the setting where the parameter space is given by a coset space G/H. We have seen that Berry's connection becomes the H-connection if the energy eigenspace we are looking at possesses the highest weight state of the unitary representation of the group G that characterizes the system. Notice also that such highest subspaces can be used to define the so called vectorial coherent states [22] for the group G. Indeed, by choosing the states $|\chi, \bar{\mu}\rangle$ as the ones belonging to a highest subspace, the states of (2.12) become the vectorial coherent states. The physical implications of this condition for the energy eigenspaces need to be investigated, but we have at least seen an interesting fact that in such cases the effective theory describing the slow variables bears an unexpected resemblance with the quantum theory on coset spaces.

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